

## Quantum complex sine-Gordon model on a half line

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**ABSTRACT:** In this paper, we examine the quantum complex sine-Gordon model on a half line. We obtain the quantum spectrum of boundary bound states using the semi-classical method of Dashen, Hasslacher and Neveu. The results are compared and found to agree with the bootstrap programme. A particle/soliton reflection factor is conjectured, which is consistent with unitarity, crossing and our semi-classical results.

**KEYWORDS:** Integrable Field Theories, Exact S-Matrix, Boundary Quantum Field Theory.

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## 1. Introduction

The complex sine-Gordon model [1, 2], appears as the simplest case in a family of homogeneous generalisations for the sine-Gordon model [3–6]. It is an integrable (1 + 1) dimensional field theory with an internal U(1) degree of freedom. It is described by a Lagrangian of the form

$$\mathcal{L}_{\text{CSG}} = \frac{1}{2} \frac{\partial u \bar{\partial} u^* + \bar{\partial} u \partial u^*}{1 - \xi^2 u u^*} - m^2 u u^* , \quad (1.1)$$

where  $u$  is a complex field and  $\xi$  is a real coupling constant. The model accepts soliton [1, 7] and breather solutions [8] which are non topological in nature. The sine-Gordon model is recovered only in the chargeless limit, in which solutions become topological objects.

In a previous paper [8] we studied the classical complex sine-Gordon theory in the bulk and on a half line. The behaviour of various solutions and their interaction with the boundary was analysed. Through the construction of conserved currents we managed to obtain a general boundary condition that preserves integrability

$$\partial_1 u = -C u \sqrt{1 - u u^*} , \quad \partial_1 u^* = -C u^* \sqrt{1 - u u^*} , \quad (1.2)$$

where both expressions are evaluated at  $x = 0$  and  $C$  is a real parameter. The full Lagrangian on a half line is

$$\mathcal{L}_{\text{CSG}} = \int_{-}^0 \infty dx \left( \frac{\partial_0 u \partial_0 u^* - \partial_1 u \partial_1 u^*}{1 - \xi^2 u u^*} - m^2 u u^* \right) + [2C \sqrt{1 - u u^*}]_{x=0} . \quad (1.3)$$

In this paper, we shall aim to extend our study of the model on the half-line to the quantum regime.

The outline of this paper is as follows: in the second section we review the most important results for the quantum complex sine-Gordon theory in the bulk.

In the third section we shall use the semi-classical stationary-phase approach which is the field theory analogue of the WKB method of quantum mechanics. A generalised version of Bohr-Sommerfeld quantisation, following the method of Dashen, Hasslacher and Neveu [12, 13] will be used to obtain the semi-classical spectrum of boundary states and their first order corrections. As we shall see, these corrections induce a finite renormalisation of both the bulk and boundary coupling constants. The former is consistent with results from the bulk case.

Following that, the bootstrap programme devised by Ghoshal and Zamolodchikov [9] will be used to construct the quantum reflection factors of particles and solitons scattering off the boundary. The CSG model has a  $Z_k$ -symmetric bulk S-matrix which corresponds exactly to the coupling constant independent parts of the Affine Toda theory S-matrix. Delius and Gandenberger [23] have investigated reflection matrices for the latter theory and their results are adapted to postulate a reflection factor for the CSG model which is consistent with the bulk scattering matrix found by Dorey and Hollowood [15], crossing, unitarity and semi-classical results, at least in the case that  $k$  is even. In particular, extra factors need to be chosen carefully with poles consistent with the existence of the boundary-bound states, found in the previous section. That this can be done consistently provides some evidence in favour of our conjecture.

The paper finishes with a few general remarks about the results of the quantum case including a brief discussion about the poles in the bootstrap method and the necessary conditions in order for them to lie within the physical strip.

## 2. Quantum complex sine-Gordon theory in the bulk

The complex sine-Gordon model as a quantum field theory in the bulk was studied relatively soon after the model's introduction. The classical treatment which showed the theory to be completely integrable and to possess soliton solutions carrying a U(1) charge, prompted researchers to look into the quantum case in the hope that the nice features of the model persisted in this limit too.

The investigation of the quantum case began with the work of de Vega and Maillet [10] in which they showed that the  $S$ -matrix is factorisable at tree level. The model remains integrable and possesses a quantised spectrum of soliton solutions. Provided that a specific counterterm which depends on the field is added to the Lagrangian, the  $S$ -matrix is also factorisable at one-loop level. In their following paper [11] they used the inverse scattering method to obtain the classical two-soliton solution and the spectrum of states using the semi-classical methods by Dashen, Hasslacher and Neveu [12, 13].

The two-loop order case was studied by Bonneau in [14] who, continuing down the path of de Vega and Maillet, showed that the theory is non-renormalisable unless a finite number of counterterms (quantum corrections) are added.

After a gap of almost ten years the quantum complex sine-Gordon case was revisited by Dorey and Hollowood [15] in the light of the theory emerging as a gauged WZW model [16]. With the semi-classical results of de Vega and Maillet as a guide, they proposed an exact  $S$ -matrix based on the demands of the bootstrap programme. We begin our review with the general form of a single soliton solution

$$u = \frac{\cos(a) \exp [im \sin(a)(\cosh(\theta)t - \sinh(\theta)x)]}{\cosh (m \cos(a)(\cosh (\theta)(x - x_0) \sinh (\theta)t))}, \quad (2.1)$$

where  $a$  is a real parameter associated with the  $U(1)$  charge, and  $\theta$  the rapidity of the solution. From the expressions for the energy and charge

$$E = \int \frac{|\partial_0 u|^2 + |\partial_1 u|^2}{1 - uu^*} + m^2 uu^* \quad , \quad Q = i \int dx \frac{u^* \partial_0 u - u \partial_0 u^*}{1 - uu^*} \quad , \quad (2.2)$$

we easily obtain for the single soliton case

$$E = \frac{4m}{\xi^2} \cos(a) \cosh(\theta) \quad , \quad Q = \frac{4}{\xi^2} \left( \text{sign}[a] \frac{\pi}{2} - a \right) \quad . \quad (2.3)$$

In their paper Dorey and Hollowood argued that it is necessary to only consider specific values for the coupling constant. Specifically they argued that the only acceptable values are

$$\xi^2 = \frac{4\pi}{k} \quad , \quad k > 1 \quad , \quad (2.4)$$

where  $k$  is an integer. This agrees with the WZW interpretation of the theory where  $k$  corresponds to the level of the  $SU(2)/U(1)$  coset model. With this constraint their proposed  $S$ -matrix reproduces the semi-classical spectrum of states derived by de Vega and Maillet. In addition they proposed that the charge is conserved if it is defined modulo  $k$ . In the following, we shall use  $k$  instead of the coupling constant  $\xi$  for reasons of simplicity.

The quantum spectrum can be found by using the Bohr-Sommerfeld quantisation rule

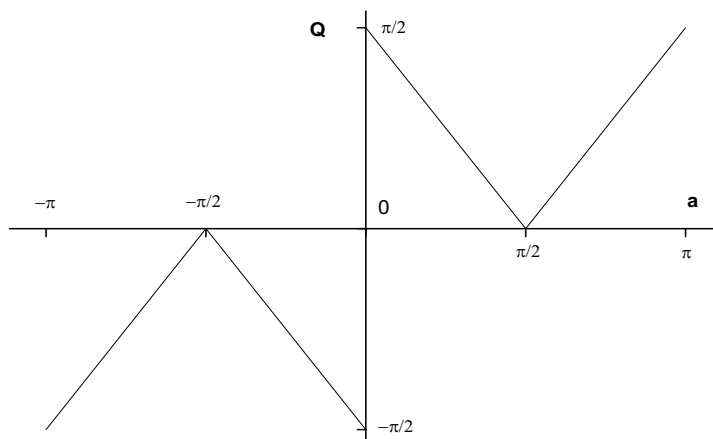
$$S(u) + E(u)\tau = 2\pi n \quad , \quad n \in \mathbb{Z} \quad , \quad (2.5)$$

where  $S$  is the action functional,  $E(u)$  the energy and  $\tau$  the period of the solution  $u$ . No topological distinction exists between the vacuum and the soliton sector, and the lowest-charge soliton may be regarded as the basic particle of the CSG theory. The static one-soliton solution is

$$u_{\text{static}} = \frac{\cos(a) \exp (im \sin(a)t)}{\cosh (m \cos(a)(x - x_0))} \quad . \quad (2.6)$$

This solution is not time-independent. It does not translate in the  $x$  direction but oscillates in a breather-like fashion. As pointed out by Ventura and Marques [17], and Montonen [18], the Bohr-Sommerfeld quantisation is equal to charge quantisation for scalar field theories enjoying a global  $U(1)$  symmetry. For the CSG case the only time dependence for the soliton solution in the rest frame is restricted to the phase i.e.  $uu^*$  does not depend on time. It is easy to show that for the static one-soliton

$$S(u) + E(u)\tau = 2\pi Q \quad , \quad (2.7)$$



**Figure 1:** Soliton charge  $Q(a)$

which in turn implies

$$Q = n. \tag{2.8}$$

This corresponds to a tower of states with ever increasing charge. However in the classical case the charge is a periodic function. In the theory  $a$  appears only in trigonometric forms, and therefore should be considered as an angle variable. This indicates that the formula for  $Q$  which has previously appeared in the literature is only true for a certain region, namely  $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$ . If one plots  $Q$  as a function of  $a$  the result is a periodic pattern (figure 1). To avoid confusion we shall consider  $a$  to lay in the region  $0 \leq a \leq \frac{\pi}{2}$ , unless stated otherwise.

As suggested by Dorey and Hollowood the charge should be defined modulo  $k$  which leads to a finite spectrum depending on  $k$

$$Q = \pm 1, \pm 2, \dots, \pm \frac{k-1}{2} \quad \text{for } k \text{ odd}, \tag{2.9}$$

$$Q = \pm 1, \pm 2, \dots, \pm \frac{k-2}{2}, \frac{k}{2} \quad \text{for } k \text{ even} . \tag{2.10}$$

From the classical expression for the charge (2.2) we have

$$Q = \frac{k}{\pi} \left( \frac{\pi}{2} - a \right) . \tag{2.11}$$

The quantisation of the charge leads to the following quantisation of  $a$ :

$$2\pi Q = 2\pi n \Leftrightarrow a = \frac{\pi}{k} \left( \frac{k}{2} - n \right) . \tag{2.12}$$

Through the quantisation of the charge parameter, the energy spectrum

$$M = \frac{k}{\pi} m \cos(a), \tag{2.13}$$

is also obtained

$$M(Q) = \frac{km}{\pi} \left| \sin \left( \frac{\pi Q}{k} \right) \right|. \tag{2.14}$$

The Bohr-Sommerfeld quantisation provides us only with the quantum spectrum up to leading order. According to de Vega and Maillet the next order corrections is achieved by a simple renormalisation of the coupling constant

$$\xi^2 \rightarrow \xi_R^2 = \frac{\xi^2}{1 - \xi^2/4\pi}. \tag{2.15}$$

or equivalently

$$k \rightarrow k_R = k - 1. \tag{2.16}$$

At this level an  $S$ -matrix may be written down that reproduces the semi-classical spectrum to leading order. In their paper Dorey and Hollowood first presented a minimal choice for the meson-soliton scattering matrix

$$S_{1Q} = F_{Q-1}(\theta) F_{Q+1}(\theta). \tag{2.17}$$

which reproduced the semi-classical behaviour and agreed with the results of deVega and Maillet. Moreover, through this they confirmed that the meson in the CSG theory can be identified with the  $Q = 1$  soliton. The function  $F_Q(\theta)$  is defined as

$$F_Q(\theta) = \frac{\sinh \left( \frac{\theta}{2} + \frac{i\pi}{2k} Q \right)}{\sinh \left( \frac{\theta}{2} - \frac{i\pi}{2k} Q \right)}. \tag{2.18}$$

$S$ -matrices constructed from products of the  $F_Q(\theta)$  automatically satisfy unitarity and analyticity constraints. With the above result as a starting point they proposed the following  $S$ -matrix for arbitrary charge

$$S_{Q_1 Q_2} = F_{Q_1 - Q_2} \left[ \prod_{n=1}^{Q_2 - 1} F_{Q_1 - Q_2 + 2n} \right]^2 F_{Q_1 + Q_2}, \tag{2.19}$$

which satisfies all the familiar restrictions and has the correct pole structure. In addition they pointed out that this is exactly the minimal  $S$ -matrix associated with the Lie algebra  $a_{k-1}$  and conjectured that for the specific choices of the coupling this  $S$ -matrix should be exact.

With the general form of the  $S$ -matrix the review of the bulk case is concluded. More details on the above results may be found in the relevant papers. Nevertheless, this presentation contain all the ingredients needed to examine the half-line case.

### 3. Semi-classical quantisation

In this section we shall attempt to build the complete spectrum of quantum boundary-bound states for the complex sine-Gordon model on a half line. The fact that the model possesses exact periodic solutions bound to the boundary makes the stationary-phase method a suitable candidate for their quantisation. The method is based on the work of Dashen, Hasslacher and Neveu [12, 13] for the semi-classical quantisation of the sine-Gordon model. In this case it appears as a generalised version of the Bohr-Sommerfeld quantisation rule

$$S_{\text{qu}} - T \frac{\partial S_{\text{qu}}}{\partial T} = 2\pi n, \tag{3.1}$$

where  $S_{\text{qu}} = S_{\text{cl}} - \Delta$ . The parameter  $\Delta$  is related to the stability angles of the stationary phase approach and produces the first order quantum corrections to the classical action. We shall use the ordinary quantisation condition for the classical action  $S_{\text{cl}}$  and then we shall calculate the quantum corrections factor

$$\Psi = \left( \Delta - T \frac{\partial \Delta}{\partial T} \right). \tag{3.2}$$

Before embarking on quantisation, we will recall a few of the classical results about the boundary spectrum from [8]. The energy of the half-line theory is given by

$$E = \int_{-\infty}^0 \left[ \frac{|\partial_0 u|^2 + |\partial_1 u|^2}{1 - uu^*} + m^2 u \right] - 2C \sqrt{1 - uu^*} \tag{3.3}$$

For a given value of  $C$ , the lowest energy configuration corresponds to the vacuum  $u = 0$ . Linearising the equations of motion and the boundary conditions (1.2) one calculates the particle reflection coefficient can be found to be

$$K(k) = \frac{ik + C}{ik - C} \tag{3.4}$$

where  $k$  is the momentum of the particle. For  $C < -m$ , one can show that the pole corresponds to a normalisable but in time exponentially growing perturbative mode showing that the vacuum is an unstable configuration. For  $C$  in this range, the boundary can emit a real soliton which effectively swaps the sign of  $C$  so that ultimately the classical field configuration returns to  $u = 0$ , but now with  $C > m$ . For  $|C| \leq m$ , it is possible to construct a classical excited boundary state by taking  $u = u_{\text{cl}}$  to be a static soliton solution of the form in (2.6), provided that the parameters are chosen to obey the equation

$$C = \pm \frac{m}{\sqrt{1 + \tan^2(a) \coth^2(m \cos(a)x_0)}} \tag{3.5}$$

There are other more complicated boundary states based on multi-soliton solutions such as breathers [8], but a thorough analysis of these is beyond the scope of the present study.

The boundary term of (1.3) preserves the  $U(1)$  charge as it only depends on  $\text{mod}(u)$ . The theory remains  $U(1)$  invariant and therefore the Bohr-Sommerfeld quantisation is

equivalent to charge quantisation on the half line case as it was for the bulk. The quantisation condition for the classical action reads

$$S_{\text{cl}}[u_{\text{cl}}] + E_{\text{cl}}[u_{\text{cl}}]T = 2\pi Q = 2\pi n . \quad (3.6)$$

with  $u_{\text{cl}}$  the static one soliton solution of (2.6) with parameters chosen to satisfy (3.5). It is a periodic solution with period  $T$  exhibiting a breather-like behaviour. It provides the perfect starting point as the simplest boundary bound state of the classical theory. The position of the centre of mass will determine the charge of the bound state as a fraction of the soliton charge in the bulk. We can calculate the charge of the static soliton through the expression

$$Q = -i \int_{-\infty}^0 dx \frac{u^* \partial_0 u - u \partial_0 u^*}{1 - uu^*} . \quad (3.7)$$

Note that this charge is conserved without the need to add a term at the boundary. This time however the integration takes place on the half line and finally yields

$$Q = \frac{k}{2\pi} (\pi - b - a) . \quad (3.8)$$

The charge now depends on the boundary parameter  $C = m \cos(b)$ , which enters the calculation through the position of the centre of mass. As a check that this formula is correct, we can set  $b = \frac{\pi}{2}$  which implies that we have the Neumann boundary condition  $\partial_x u = 0$ , thus placing the soliton at exactly  $x = 0$ . The charge is then exactly half of the equivalent charge of the soliton in the bulk (2.3) as expected. The quantisation condition now reads

$$Q = \frac{k}{2\pi} (\pi - b - a) = n , \quad (3.9)$$

or in terms of the charge parameter

$$a = \pi - b - \frac{2\pi n}{k} . \quad (3.10)$$

The quantisation of  $a$  provides us with the first approximation of the semi-classical spectrum

$$E_n = \frac{km}{2\pi} \cos(a) = \frac{km}{2\pi} \cos\left(\pi - b - \frac{2\pi n}{k}\right) . \quad (3.11)$$

Having a general formula for the energy, it is useful to calculate the energy difference between two adjacent states. We shall use this in the following section where we shall compare it with the corresponding bootstrap result. For reasons of simplicity, we shall assume that the values of the parameters are such that the cosine is positive for both states as is their difference so that we can ignore any modulus appearing. The energy difference is then written

$$E_{n+1} - E_n = \frac{km}{2\pi} \left( \cos\left(\pi - b - \frac{2\pi(n+1)}{k}\right) - \cos\left(\pi - b - \frac{2\pi n}{k}\right) \right) , \quad (3.12)$$



which after some manipulation simplifies to

$$E_{n+1} - E_n = \frac{km}{\pi} \cos\left(\frac{\pi}{2} - \frac{\pi}{k}\right) \cos\left(\frac{\pi(2n+1)}{k} - \frac{\pi}{2} + b\right). \quad (3.13)$$

We shall return to this result at the end of the next section, after we have obtained a relevant expression from the bootstrap programme.

Having obtained the leading order quantisation of the boundary spectrum, we calculate the first-order correction to this. The step consists of calculating the quantity  $\Delta$

$$\Delta = -S_{\text{ct}} + \sum_{i=0}^{\infty} \frac{1}{2} \hbar v_i, \quad (3.14)$$

where  $v_i$  are the stability angles and  $S_{\text{ct}}$  suitably selected counter terms to cancel any infinities that arise. We shall first calculate the sum over the stability angles, and introduce counter terms later as required. Following the standard procedure, we regulate the theory by placing it in a box. The theory is already bounded on the right from the original boundary, therefore another boundary should be introduced restricting the system to the finite interval  $[-L, 0]$ . Placing the model in a box has the effect of discretising its energy levels. Afterwards, we shall take the limit  $L \rightarrow -\infty$  to recover the original system. A simple and appropriate choice for boundary conditions at  $x = -L$  is to take the Dirichlet boundary conditions  $u = 0$ , which reproduce the correct behaviour of  $u$  at infinity when we take the limit  $L \rightarrow -\infty$ .

The stability angles are obtained by solving the linearised stability equation about a given classical solution. In our case we perturb around the static one-soliton solution. Once the solutions  $\chi(x, t)$  for the stability equation are found then the stability angles can be calculated from periodicity demands

$$\chi_i(x, t + T) = e^{iv_i} \chi_i(x, t), \quad (3.15)$$

Instead of solving directly the stability equation we can follow the method of Corrigan and Delius to calculate the sum of the stability angles through the reflection factors. We begin with the classical two-soliton solution of the CSG model which satisfies the stability equation. By fixing the free parameters we can make one of the solitons  $S_1$  static by taking  $\delta_1 = 1$  and the other  $S_2$  very small by taking the charge parameter  $a_2$  close to zero. Effectively we are left with a small perturbation around a static one-soliton background. This is exactly the same method that Dashen, Hasslacher and Neveu followed to calculate the stability angles of the sine-Gordon model. At infinity the static soliton is practically zero whilst the perturbation appears as plane waves

$$\chi(x, t) = e^{-i\omega t} (e^{ik_s x} + R_s e^{-ik_s x}). \quad (3.16)$$

The reflection factor is found to be

$$R_s = -\frac{(\cos(b - i\theta))}{(\cos(b + i\theta))} \frac{(\sin(a + i\theta) - 1)}{(\sin(a - i\theta) - 1)}, \quad (3.17)$$

where we have set  $m = 1$  without loss of generality. The rapidity  $\theta$  which has been used is related to the momentum through  $k = \sinh(\theta)$ . The parameters  $a$  and  $b$  are the familiar charge and boundary parameters. From the above expression for  $\chi(x, t)$  and (3.15) we can substitute

$$\Delta = \frac{1}{2} \sum_i v_i = \frac{T}{2} \sum_i \omega_i, \quad (3.18)$$

where  $\omega_i^2 = k_i^2 + 1$  and the counter terms have been neglected. Placing the system in a box allows for discrete values of  $k$ , which should not be confused with the coupling constant. From  $\Delta$  we must now subtract the vacuum contributions

$$\Delta = \frac{T}{2} \sum_i \sqrt{k_{s,i}^2 + 1} - \sqrt{k_{0,i}^2 + 1}. \quad (3.19)$$

From the Dirichlet boundary conditions we can obtain the following equation relating the discrete momenta with the reflection factors

$$e^{-2ik_s L} = \frac{(\cos(b - i\theta))}{(\cos(b + i\theta))} \frac{(\sin(a + i\theta) - 1)}{(\sin(a - i\theta) - 1)}. \quad (3.20)$$

A similar equation exists for the fluctuations around the vacuum

$$e^{-2ik_0 L} = -\frac{i \sinh(\theta) + \cos(b)}{i \sinh(\theta) - \cos(b)}. \quad (3.21)$$

Using the same argument as Corrigan and Delius, we can define a function  $\kappa(k_0)$  such that for large  $k$  we may write

$$k_s = k_0 + \frac{\kappa(k_0)}{L}, \quad (3.22)$$

where the index  $i$  has been suppressed. This is possible since in the limit  $\theta \rightarrow +\infty$  and taking into account the general quantisation condition (3.9), both reflection factors  $R_s$  and  $R_0$  are equal. Through the difference  $k_s - k_0$  a function  $\kappa(\theta)$  may be defined using the ratio of the corresponding reflection factors

$$e^{-2i\kappa(\theta)} = -\frac{(\sin(a + i\theta) - 1)}{(\sin(a - i\theta) - 1)} \frac{(\sin(b + i\theta) - 1)}{(\sin(b - i\theta) + 1)}. \quad (3.23)$$

We can now calculate  $\Delta$  in terms of  $\kappa$ . We can substitute (3.22) in (3.19) and then expand the expression in terms of  $L$ . Keeping only the leading term of the expansion we end up with

$$\Delta \sim \frac{T}{2L} \sum_i \frac{k_i^{(0)} \kappa(k_i^{(0)})}{\sqrt{(k_i^{(0)})^2 + 1}}, \quad (3.24)$$

which in the limit  $L \rightarrow +\infty$  can be substitute with the integral form

$$\Delta = \frac{T}{2\pi} \int_0^{=\infty} dk \frac{k}{\sqrt{k^2 + 1}} \kappa(k). \quad (3.25)$$

The calculation is greatly simplified if we change variables to  $\theta$

$$\Delta = \frac{T}{2} \int_0^{+\infty} d\theta \sinh(\theta) \kappa(\theta) . \quad (3.26)$$

The integral is divergent but we can introduce specially chosen counter-terms to obtain a finite result. We begin with an integration by parts

$$\Delta = \frac{T}{2\pi} \left( [\kappa \cosh(\theta)]_0^{+\infty} - \int_0^{+\infty} \cosh(\theta) \frac{d\kappa}{d\theta} \right) . \quad (3.27)$$

The first term is not divergent since the function  $\kappa$  approaches zero (through the quantisation condition) as  $\theta$  goes to infinity. In this limit the combination  $\kappa(\theta) \cosh(\theta)$  is zero. In addition,  $\kappa(0) = 0$  so the first term in (3.27) vanishes. The second term is however divergent and counter terms have to be introduced to cancel infinities. The latter appear in the same fashion as the logarithmic divergencies in the bulk which are tackled through normal ordering. With some straightforward manipulation the derivative term yields

$$\frac{d\kappa}{d\theta} = -\frac{\cos(b)}{\cosh(\theta) - \sin(b)} + \frac{\cos(a)}{\cosh(\theta) - \sin(a)} . \quad (3.28)$$

The two terms are almost identical and both divergent. A logical choice of counter-terms seems to be

$$-\frac{\cos(b)}{\cosh(\theta) + 1} + \frac{\cos(a)}{\cosh(\theta) + 1} , \quad (3.29)$$

where the first removes the divergence associated with the boundary and the second with the one in the bulk. The complete expression to be calculated is now

$$\Delta = - \int_0^{+\infty} d\theta \cosh(\theta) \left( \frac{\cos(a)}{\cosh(\theta) - \sin(a)} - \frac{\cos(a)}{\cosh(\theta) + 1} - \frac{\cos(b)}{\cosh(\theta) - \sin(b)} + \frac{\cos(b)}{\cosh(\theta) + 1} \right) ,$$

which finally yields

$$\Delta = \frac{T}{2\pi} \left( -\cos(a) + \cos(b) + b \sin(b) + \frac{\pi}{2} \sin(b) - a \sin(a) - \frac{\pi}{2} \sin(a) \right) . \quad (3.30)$$

Having determined the form of  $\Delta$  we are now in a position to calculate the corrections to the classical Bohr-Sommerfeld rule. The expression of  $\Delta$  depends on the period  $T$  through the charge parameter  $a$

$$\sin(a) = \frac{2\pi}{T} . \quad (3.31)$$

The correction term of (3.2) is easily calculated

$$\Delta - T \frac{\partial \Delta}{\partial T} = -a - \frac{\pi}{2} . \quad (3.32)$$

With all the necessary parts calculated, the generalised Bohr-Sommerfeld quantisation condition finally reads

$$k_r(\pi - b_r - a) = 2\pi n , \quad (3.33)$$

where

$$k_r = k - 1, \quad b_r = \frac{kb - \frac{3\pi}{2}}{k - 1}. \quad (3.34)$$

The form of the new quantisation condition is exactly the same as the first approximation of (3.9), only with a redefinition for the boundary and coupling constants. The shift in the coupling constant is to be expected. In the bulk case the first order corrections amount to a simple shift in  $k$  (2.16) as pointed by de Vega and Maillet [19]. Our result is consistent with this.

In addition to  $k$ , the boundary constant has to be renormalised as well. It is not clear why this renormalisation is needed or whether it should appear at all. In a paper examining the closely related  $a_{k-1}$  theory, Penati and Zanon [20] argued that renormalisation of boundary parameters has to be introduced in certain models to ensure integrability at the quantum level. The renormalisation of the coupling constant and its significance remains one of the open questions for the quantum CSG theory.

#### 4. The bootstrap method

In the previous section we constructed the quantum spectrum using the semi-classical stationary-phase method. Although the results are not exact, they provide us with an accurate picture of the set of states. The same spectrum can be obtained using the completely different approach of the bootstrap method, based on the pioneering work of Cherednik [21], Ghoshal and Zamolodchikov [9], and Fring and Köberle [22]. The idea behind this method is to construct the reflection factors through the boundary bootstrap relation, and through the poles therein to identify boundary bound states. The process is analogue to the bulk case where the existence of bound states is indicated by poles found in the  $S$ -matrix.

In this section we shall conjecture the form of the full quantum reflection factor for the charge one particle of the theory reflecting off the unexcited (vacuum) boundary. This is constrained by the principles of analyticity, unitarity and crossing symmetry, together with the boundary Yang-Baxter equation. Furthermore the factor should contain a pole in the physical strip consistent with the existence of a boundary bound state found above. The reflection amplitudes for higher charged solitons, both reflecting off the vacuum and excited boundaries, can be deduced from this basic amplitude by the repeated use of the reflection bootstrap equations and boundary bootstrap equations respectively. These contain poles consistent with the spectrum of bound states found in the previous section. Also the classical limit of our conjecture coincides with the classical results (3.4), (3.17). Finally we compare our results with existing results [9, 25], for the  $q$ -state Potts models whose bulk  $S$ -matrix coincides with that of the CSG for  $q = 2$  (the Ising model) and  $q = 3$ , and find agreement in those cases that correspond to diagonal scattering.

Whilst our conjectured reflection factors are highly restricted by the requirements of analyticity, unitarity and crossing symmetry, the correct classical limit, and some of the correct pole structure they might not be pinned down completely by these requirements, and our conjecture is in some sense a ‘minimal’ solution to the constraints. Furthermore

we have not attempted a complete explanation of the physical strip poles appearing in the reflection amplitudes, some of which might be explained by Coleman-Thun like on-shell diagrams and others by more exotic boundary bound states as is the case for Toda theories [23].

#### 4.1 Quantum reflection factor for the CSG particle

We begin our attempt to introduce a suitable reflection matrix  $K_1$  for the CSG particle with the assumption that it should be made out of  $F$  factors that were defined in (2.18). It is a natural selection as both the boundary Yang-Baxter equation and the bootstrap equations relate reflection matrices with the  $S$ -matrix which is expressed only in terms of such functions. There is another advantage to this choice. Unitarity, real analyticity and  $2\pi i$ -periodicity requirements are automatically satisfied if  $K$  appears as a product of such factors. Some of the most important properties enjoyed by these functions include the following

$$F_Q(\theta)F_Q(-\theta) = 1, \quad F_{2Q}(2\theta) = -F_Q(\theta)F_{Q+k}(\theta), \quad F_Q(\theta + i\pi) = -F_{Q+k}(\theta).$$

The first of the above is responsible for the fulfillment of the unitarity requirement. The remaining equations demonstrate basic transformations between rapidity and charge and will be used in the bootstrap and crossing symmetry equations.

The spectrum of the model is only degenerate under charge conjugation. Given that charge is conserved by our choice of boundary conditions, we shall assume that the reflection scattering matrix is diagonal, and accordingly we shall denote the scattering matrix of the charge  $a$  soliton/particle by  $K_a$ . Since the model is symmetric under charge conjugation we shall take  $K_a = K_{\bar{a}}$  where  $\bar{a}$  denotes the antiparticle of particle  $a$ . For the  $Z_k$  symmetric model this can be written  $K_a = K_{k-a}$ . Since both  $K_a$  and the bulk  $S$ -matrix are diagonal the boundary Yang-Baxter equation

$$K_a(\theta_a)S_{ab}(\theta_a + \theta_b)K_b(\theta_b)S_{ab}(\theta_a - \theta_b) = S_{ab}(\theta_a - \theta_b)K_b(\theta_b)S_{ab}(\theta_a + \theta_b)K_a(\theta_a),$$

where  $\theta_a, \theta_b, \theta_c$  are rapidities associated with a fusion process  $a + b \rightarrow c$ , is trivially satisfied. In their paper Dorey and Hollowood identified the CSG particle with the  $Q = 1$  soliton. In this context the first real constraint for  $K$  comes from the crossing symmetry relation which for our case is written as

$$K_1(\theta)K_{\bar{1}}(\theta + i\pi) = S_{1,1}(2\theta), \tag{4.1}$$

where  $K_{\bar{1}}$  is the reflection of the antiparticle and  $S_{1,1}$  is the two particle scattering matrix. The CSG  $S$ -matrix (2.19) is identical to the minimal  $a_{k-1}$   $S$ -matrix which in turn can be recovered from the  $a_{k-1}^{(1)}$  Affine Toda field theory (ATFT) when the parts involving the coupling constant are omitted. It is therefore reasonable to build our reflection matrix based on the proposed form for the particle reflection matrix of the boundary  $a_{k-1}^{(1)}$  ATFT theory. In their paper Delius and Gandenberger [23] present a general form for the particle reflection matrix of the  $a_{k-1}^{(1)}$  ATFT. As in the  $S$ -matrix case, the reflection matrix is a product of two parts, one of which depends on the coupling constant and the other is

independent of the coupling constant. Each part satisfies the bootstrap independently so we can recover a  $K$ -matrix for our model by simply ignoring the coupling dependent pieces. Henceforth we shall adopt the block notation notation  $(x) = F_x(\theta)$ . Ignoring the parts involving the coupling constant in the ATFT reflection factor, the remaining factors

$$K_n^{\text{base}} = \prod_{c=1}^n (c-1)(c-k), \quad n = 1..(k-1), \quad (4.2)$$

constitute a complete set satisfying unitarity, the crossing-symmetry condition, charge conjugation symmetry, (4.1) as well as the reflection bootstrap equation

$$K_c^{\text{base}}(\theta_c) = K_a^{\text{base}}(\theta_a) K_b^{\text{base}}(\theta_b) S_{ab}(\theta_a + \theta_b). \quad (4.3)$$

However this reflection factor cannot be the full story;  $K^{\text{base}}$  does not contain any poles which can be related to boundary-bound states discussed in previous sections. To rectify this, we need to multiply the reflection factor by an additional factor  $Z_a(\theta)$ , so that

$$K_a(\theta) = Z_a(\theta) K_a^{\text{base}}(\theta), \quad (4.4)$$

where  $Z_a(\theta)$  contains poles in the physical strip corresponding to boundary bound states. In order that the new reflection matrix be consistent with unitarity, crossing symmetry, charge conjugation symmetry and the reflection bootstrap we demand that

$$Z_a(\theta) Z_a(-\theta) = 1, \quad Z_a(\theta) Z_a(\theta + i\pi) = 1, \quad Z_a(\theta) = Z_{k-a}(\theta), \quad Z_c(\theta_c) = Z_a(\theta_a) Z_b(\theta_b) \quad (4.5)$$

where as above  $\theta_a, \theta_b, \theta_c$  are the rapidities associated with a fusion process  $a+b \rightarrow c$ . (In [9] and [23],  $Z_a$  are called CDD ambiguities, but strictly speaking CDD factors should not contain poles in the physical strip, while the factors  $Z_a$  are specifically designed to contain such poles, so we shall refrain from using this nomenclature.)

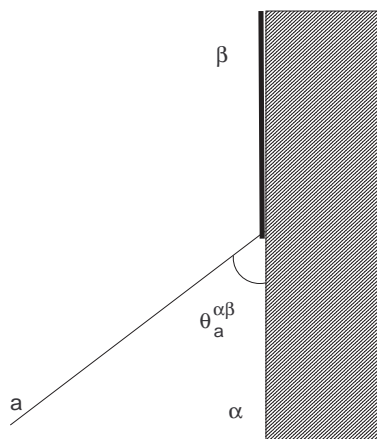
We begin our search for a suitable factors  $Z_a$  by determining where the poles corresponding to the bound states we have described should lie. Consider the process where the unexcited boundary becomes excited by the absorption of a charged soliton, as described in (figure 2). During the process both energy and charge are conserved. We begin with the charge conservation. Far away from the boundary the soliton (particle) behaves as in the theory in the bulk. Its' charge is equal to the normal soliton charge  $Q_1 = \frac{k}{\pi}(\frac{\pi}{2} - a_1)$ . After the formation of the bound state the charge  $Q_2$  of the boundary is given by the formula (3.8). Equating these yields

$$\left(\frac{\pi}{2} - a_1\right) = \frac{1}{2}(\pi - b - a_2). \quad (4.6)$$

Using the same arguments we can write down an equation describing the conservation of energy

$$4m \cos(a_1) \cosh(\theta) - 2m \cos(b) = 2m \cos(a_2), \quad (4.7)$$

where the  $-2m \cos(b)$  term is the boundary energy contribution when the field is zero. For simplicity, we have assumed in the above equations that  $0 < a_i < \pi/2$  and that



**Figure 2:** The boundary bound state.

$a_2 < b < \pi - a_2$ ; the generalisation of these equations to other ranges of the parameters is straightforward. From (4.6) and (4.7) we can determine the rapidity  $\theta$  at which the boundary bound state is formed

$$\theta_n^{(0,n)} = i\frac{\pi}{k} (n + B) \text{ where } B = \frac{k}{\pi} b - \frac{k}{2} . \tag{4.8}$$

For the above relation the quantisation condition (3.9) was also used. It is worth noting that in deriving this equation we assumed that the charge of the incoming particle was positive, or in terms of  $n$  that  $0 < n < [k/2]$  where  $[x]$  stands for the integer part of  $x$ .

Real analyticity and unitarity conditions once again prompt us to construct the  $Z_a$  factors out of block functions, so that the former are satisfied automatically. In order for the crossing relation to be satisfied any block factor  $(x)$  should be accompanied by the charge conjugate factor  $(k - x)$ . Delius and Gandenberger showed for such a combination the bootstrap closes. Now that we have determined where the pole should be, we need to express it in block notation. Since  $(x)$  has a pole in the denominator at  $\theta = i\frac{\pi}{k}x$ , from (4.8) we might expect  $Z_n$  to contain the factor  $(B + n)(k - B - n)$ , for  $0 < n < [k/2]$ . The  $Z_n$  for larger  $n$  are determined by the charge conjugation symmetry  $Z_n = Z_{k-n}$ . The factors  $Z_n$  might contain other factors besides these and indeed we will find that this is the case. The number of blocks in  $K_n$  increases as  $n$  increases towards  $\frac{k}{2}$  and then decreases as it approaches  $k - 1$ , consistent with our demand that the theory is  $Z_k$  symmetric so that  $K_{k+1} = K_1$ , and that under charge conjugation we expect  $K_n = K_{k-n}$ .

To proceed we can decide on a ‘minimal choice’ for  $K_1$ , that is we choose the factor  $Z_1$  to consist of the first pole and its’ charge conjugate counterpart. The full reflection factor then reads

$$K_1^{(0)} = (1 + B)(k - 1 - B)(1 - k) . \tag{4.9}$$

The superscript denotes the boundary excitation state. In this particular case  $K_1^{(0)}$  describes the reflection of a soliton of charge  $Q = 1$  from an unexcited boundary. In the present analysis we shall only consider poles in the factors  $Z_n$  and not in  $K_n^{\text{base}}$ . The pole associated with the  $(1 + B)$  factor corresponds to the lowest boundary bound state with energy

$$E_b^{(1)} = \frac{km}{2\pi} \cos\left(\pi - b - \frac{2\pi}{k}\right). \quad (4.10)$$

which is formed when a soliton of charge  $Q = 1$  (which is the particle of the theory) fuses with the boundary. The conjugate term  $(k - 1 - B)$  also has a pole but does not lie within the physical strip  $\Im(\theta) \in (0, \frac{\pi}{2})$ . Note that in the classical limit  $k \rightarrow \infty$  the reflection factor

$$K_1^{(0)} \rightarrow \frac{i \sinh(\theta) + \cos(b)}{i \sinh(\theta) - \cos(b)} \quad (4.11)$$

in agreement with the identification of the charge 1 soliton as a particle and the classical particle reflection factor (3.4).

With  $K_1^{(0)}$  as a starting point we can construct the whole set of  $K_n^{(0)}$  factors by using the reflection bootstrap (4.3). Later we shall also apply the boundary bootstrap

$$K_b^{(\beta)} = S_{ab}(\theta_b - \theta_a^{\alpha\beta}) K_b^{(\alpha)} S_{ab}(\theta_b + \theta_a^{\alpha\beta}), \quad (4.12)$$

to each of them to obtain the corresponding reflection matrices from the excited boundary. In both cases new block factors are generated carrying poles which indicate new bound states. If our choice for the factor  $Z_a$  in (4.9) is correct we expect the bootstrap to close, i.e. to end up with a finite spectrum of states which is repeated after  $k$  steps. Pictorially this can be seen in (figure 3)

Let us begin by calculating the factors  $K_n^{(0)}$  describing reflection off the unexcited boundary. For instance we can calculate the reflection factor of a charge  $Q = 2$  soliton bouncing off the unexcited boundary using the equation

$$K_2^{(0)}(\theta) = K_1^{(0)}\left(\theta - \frac{i\pi}{k}\right) K_1^{(0)}\left(\theta + \frac{i\pi}{k}\right) S_{1,1}(2\theta). \quad (4.13)$$

Substituting  $K_1^{(0)}$  from (4.9) and  $S_{1,1}$  from (2.19) we get

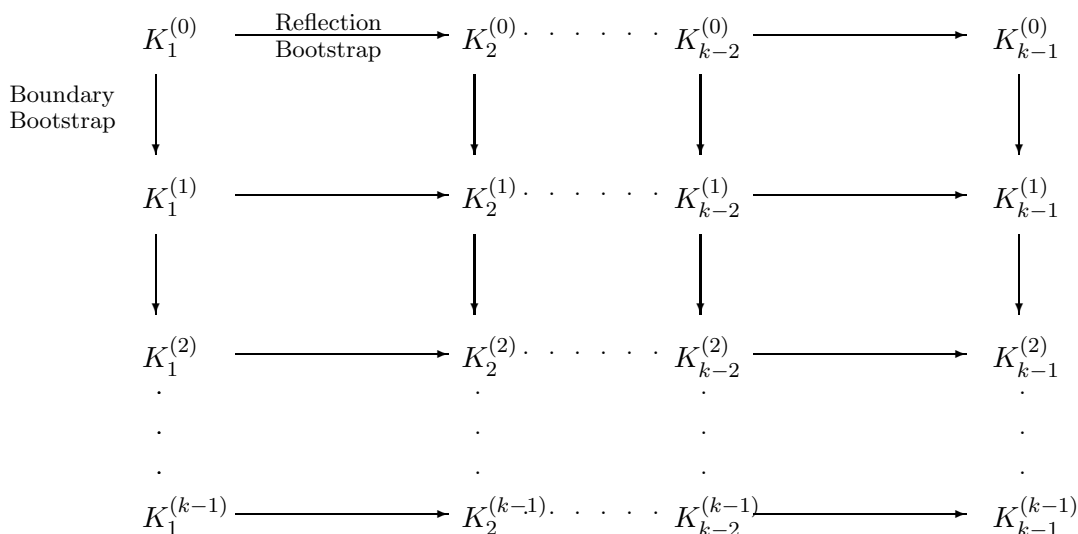
$$K_2^{(0)} = (2 + B)(k - 2 - B)(B)(k - B)(1 - k)(1)(2 - k). \quad (4.14)$$

By iterated use of the reflection bootstrap equation we arrive at the formula

$$K_n^{(0)} = K_n^{\text{base}} \prod_{i=0}^{n-1} (B + 2 - n + 2i)(k - B - 2 + n - 2i) \quad (4.15)$$

Note that this function is a product of blocks of the general form  $(c)(k - c)$  with which we started. In particular note that the factor  $Z_n$  in  $K_n^{(0)}$  always involves the block pair  $(n + B)(k - n - B)$  as expected.





**Figure 3:** The Bootstrap Programme.

For even  $k$  it is easy to show that the bootstrap closes, and we conjecture that (4.15) are the correct quantum reflection factors in this case. In particular we have that  $K_a = K_{k-a}$  as desired. This follows because the terms in (4.15) of the form  $(B + 2 - n + 2i)$  can cancel with those of the form  $(k - B - 2 + n - 2j)$  for appropriate  $i, j$  and for  $k$  even, using the properties that  $(x) = 1/(-x)$ . However this is not possible in the case that  $k$  is odd. In this case our conjecture for  $K_1^{(0)}$  needs to be modified. A possibility which is consistent with charge conjugation symmetry, as well as unitary, crossing symmetry and the reflection bootstrap is to take

$$K_1^{(0)} = (1 + B)(k - 1 - B)(B + c)(k - c - B)(1 - k) \tag{4.16}$$

where  $c$  is any even integer. Unfortunately, an additional factor  $(B + c)(k - c - B)$  is constrained to be 1 in the classical limit  $k \rightarrow \infty$ , since our previous conjecture (4.9) already had the correct classical limit, and there is no choice of  $c$  which gives this classical limit. Thus we have been unable to find a suitable reflection factor in the case that  $k$  is odd, and in what follows we shall take  $k$  to be even.

### 4.2 The boundary bootstrap

In the previous section we proposed a suitable expression for the  $K_1^{(0)}$  reflection factor for  $k$  even, and saw how through the reflection bootstrap all the  $K_n^{(0)}$  factors can be obtained. We now turn to the boundary bootstrap to construct the boundary reflection factors of solitons bouncing off an excited boundary. This will allow us to find a formula for the reflection matrix  $K_m^{(n)}$  describing how a particle of charge  $m$  scatters off a boundary of charge  $n$ . We shall not attempt to give a complete analysis of all the poles appearing in the expression for  $K_m^{(n)}$  but we will verify that it does contain the poles that we would expect from our semi-classical analysis. In particular we will show that the energy difference between adjacent boundary bound states is consistent with the semi-classical formula (3.13).

Let us first derive the reflection matrix  $K_1^{(n)}$  describing how a particle scatters off a boundary of charge  $n$ . We begin with the boundary bootstrap equation (4.12) for  $K_1^{(1)}$  which is

$$K_1^{(1)} = K_1^{(0)} S_{1,1}(\theta + \theta_1^{(0,1)}) S_{1,1}(\theta - \theta_1^{(0,1)}), \quad (4.17)$$

where  $\theta_1^{(0,1)} = i\frac{\pi}{k}(1+B)$  denotes the pole in  $K_1^{(0)}$ . The product of  $S$ -matrices in the above relation is equal to a shift in their charge. In general the following relation holds for any  $\psi$ .

$$S_{1,1}(\theta + i\psi) S_{1,1}(\theta - i\psi) = \left(2 + \frac{k}{\pi}\psi\right) \left(2 - \frac{k}{\pi}\psi\right). \quad (4.18)$$

Putting everything together yields

$$K_1^{(1)} = (1+B)(k-1-B)(3+B)(1-B)(1-k). \quad (4.19)$$

We see through this procedure a new pole appears from the block  $(3+B)$  corresponding to the absorption of a charge  $Q=1$  particle into the charge  $Q_b=1$  boundary. This can then be used as an input to find  $K_1^{(2)}$  from the boundary bootstrap,

$$K_1^{(2)} = (k-1-B)(3+B)(1-B)(5+B)(1-k), \quad (4.20)$$

where a cancellation has already taken place. A new pole comes from the block factor  $(5+B)$  which will be used in the next step. The expression for  $K_1^{(3)}$  is

$$K_1^{(3)} = (k-1-B)(1-B)(5+B)(7+B)(1-k), \quad (4.21)$$

where new factor  $(-3-B)$  has cancelled with the block factor  $(3+B)$  and the factor  $(7+B)$  has introduced a new pole. This procedure will not continue indefinitely. After  $k$  steps we expect to return to original form of  $K_1^{(0)}$ . However it is clear that the  $K_1^{(n)}$  reflection matrix will have a pole indicated by the block factor  $(2n+1+B)$  at  $\psi_n = \frac{\pi}{k}(2n+1+B)$ . Having a general formula about the  $n$ -th pole allows us to write down a recursive relation for the energies of the bound states. We begin with the energy of the first bound state. The difference between the first excited boundary state and the non-excited boundary is

$$E_1 - E_0 = A \cos(\psi_0). \quad (4.22)$$

The right hand side is equal to the mass of the incoming particle that binds with the boundary at a fixed angle  $\theta = i\psi_0$ . The parameter  $A$  is related to the mass of the particle and will be determined from the comparison with the semi-classical results. The formula may be used recursively to finally yield

$$E_{n+1} - E_n = A \cos(\psi_n) = A \cos\left(\frac{\pi}{k}(2n+1+B)\right). \quad (4.23)$$

This formula should be compared with the one derived from the semi-classical approach (3.13). Both describe the exact same energy gaps between two bound states. The arbitrary parameter  $A$  in (4.23) can be read directly from (3.13)

$$A = \frac{km}{\pi} \cos\left(\frac{\pi}{2} - \frac{\pi}{k}\right) = \frac{km}{\pi} \sin\left(\frac{\pi}{k}\right) \quad (4.24)$$

which is equal to the energy of the particle. In the limit that  $k \rightarrow \infty$  (the classical limit) the soliton becomes infinitesimally small and the boundary-bound states spectrum become continuous.

We now give a general formula for  $K_m^{(n)}$ . These can be obtained from  $K_m^{(0)}$  using the boundary bootstrap relation

$$K_m^{(n)} = S_{n,m}(\theta + \theta_n^{(0,n)})S_{n,m}(\theta - \theta_n^{(0,n)})K_m^{(0)} \tag{4.25}$$

Explicitly we find that

$$\begin{aligned} K_m^{(n)} &= (B + 2n - m)(-m - B)(2n + m + B)(m - B) \\ &\times \prod_{i=1}^{m-1} (2n + 2i - m + B)^2 (2i - m - B)^2 \prod_{c=1}^m (c - 1)(c - k) \\ &\times \prod_{j=0}^{m-1} (B + 2 - m + 2j)(k - B - 2 + m - 2j) \end{aligned} \tag{4.26}$$

Note that if one repeats the analysis of charge conservation and energy conservation for a particle/soliton of charge  $m$  being absorbed by the boundary, but now in the case that the boundary is excited with initial charge  $n$  one discovers that one would expect a pole at

$$\theta_m^{(n,n+m)} = i\frac{\pi}{k} (2n + m + B) . \tag{4.27}$$

and indeed our solution exhibits such a pole. Also, note that both  $K_m^{(n)}$  and  $K_m^{(0)}$  satisfy the same reflection bootstrap equations, so that their ratio

$$\frac{K_m^{(n)}}{K_m^{(0)}} = Z_m = S_{n,m}(\theta + \theta_n^{(0,n)})S_{n,m}(\theta - \theta_n^{(0,n)}) \tag{4.28}$$

satisfy the equations (4.5). This is exactly the form of the one-parameter solutions to these equations discussed in [26].

We conclude this section with a brief discussion about the poles appearing in the reflection matrix. We expect the terms coming from  $K_n^{\text{base}}$  which are independent of the boundary parameter  $B$  in (4.2) to be explained in terms of a on-shell triangle diagram as is the case for the  $a_n^{(1)}$  affine Toda theory. A full discussion can be found in [23]. The remaining poles which depend on  $B$  are associated with boundary-bound states. The full set of poles generated from the bootstrap programme come from the general blocks  $(n + B)$  and  $(k - n - B)$  which have poles at

$$-i\theta_n = \psi_n = \frac{\pi}{k}n - \frac{\pi}{2} + b \text{ and } -i\theta'_n = \psi'_n = \pi - \left(\frac{\pi}{k}n - \frac{\pi}{2} + b\right) \tag{4.29}$$

From the forms above we can see that if  $\psi_n$  lies in the physical strip  $(0, \frac{\pi}{2})$  then  $\psi'_n$  does not, and vice versa. Assuming that the pole is at  $\psi_n$  then it lies in the physical strip if

$$\frac{\pi}{2} > \psi_n > 0 \Leftrightarrow 1 > \frac{n}{k} + \frac{b}{\pi} > \frac{1}{2} \tag{4.30}$$

As soon the above condition is no longer true then the pole appears in the conjugate block at  $\psi'_n$ . This however does not alter any of our results. From (4.23) we can see that  $\psi_n$  or  $\psi'_n$  appear in the argument of the cosine so both correspond to the same absolute energy difference between two states.

### 4.3 Results for $k = 2$ and $k = 3$

For  $k = 2$  and  $k = 3$  the bulk  $S$ -matrix of the complex sine-Gordon model coincides with that of the  $q$ -state Potts model for  $q = 2$  (Ising model) and  $q = 3$  respectively, and can be compared with the results [9, 25]. For  $k = 2$ , which corresponds to the Ising model, the CSG model has only one particle whose reflection factor is given by

$$K_1^{(0)} = (1 + B)(1 - B)(-1) = i \tanh\left(\frac{\theta}{2} - i\frac{\pi}{4}\right) \frac{i \sinh(\theta) - \sin(b)}{i \sinh(\theta) + \sin(b)} \quad (4.31)$$

This can be compared with the reflection factor for the Ising model given in [9]:

$$R_h(\theta) = i \tanh\left(\frac{\theta}{2} - i\frac{\pi}{4}\right) \frac{i \sinh(\theta) - \kappa}{i \sinh(\theta) + \kappa} \quad (4.32)$$

where  $\kappa = 1 - h^2/2m$  and  $h$  is the boundary magnetic field. These two expressions are identical if we identify  $\kappa = \sin(b)$ , or in terms of the boundary coupling  $C$

$$\pm \sqrt{1 - C^2} = 1 - \frac{h^2}{2m}. \quad (4.33)$$

Free boundary conditions in the Ising model correspond to setting the boundary magnetic field  $h = 0$ . In the CSG model, this corresponds to turning off the boundary potential, i.e. putting  $C = 0$  which results in Neumann conditions  $\partial_x u = 0$ . Fixed boundary condition in the Ising model correspond to the limit  $h \rightarrow \infty$ . This corresponds to the limit  $C \rightarrow \infty$  in the CSG model, which results in the Dirichlet-like condition  $u = 0$ .

Although reflection factors for the  $q = 3$  Potts model have been found [25], we are unable to compare these with our results, since we do not have a conjecture for the reflection factor in the case that  $k$  is odd. Nonetheless we might be able to use these results as a clue how the case for odd  $k$  works. Chim gives results for fixed and free boundary conditions. His result for fixed boundary conditions can be written

$$R_f(\theta) = (-2) = K_1^{\text{base}}(\theta) \quad (4.34)$$

so in this case  $Z_1 = 1$ . Chim's solution for free boundary conditions corresponds to non-diagonal scattering in which the particle 1 is reflected from the boundary as the antiparticle  $\bar{1}$ . This cannot correspond to any of the charge-preserving integrable boundary conditions for the CSG considered in this paper.

## 5. Discussion

We used semi-classical methods to obtain the boundary spectrum of the CSG model. Having incorporated the unitary and crossing constraints, we found that we needed to adjust

our simplest guess for the reflection factor of a particle by an extra factor, in order that it exhibited a pole corresponding to process of absorption of a particle by the boundary. Reflection factors for higher charge solitons, and for particles and solitons scattering off excited boundaries were obtained by iterating the reflection bootstrap and boundary bootstrap respectively. These formed a consistent closed set under the bootstrap, at least in the cases that the coupling constant  $k$  was taken to be even. Although a complete survey of the poles appearing in these reflection factors was not conducted, the position of poles amongst all the reflection factors appears to be consistent with the boundary bound spectrum derived semi-classically, suggesting that this might be the complete set. Whether a consistent reflection factor for the cases where  $k$  is odd can be found is an interesting question for future research.

One final point to be made is that although we have used for our comparison equations (3.9) and (3.13) which correspond to the first approximation in the semi-classical approach, we can extend our results to agree with the first order corrections by simply substituting everywhere  $k = k_r$  and  $b = b_r$ . The redefinition of both parameters change nothing in the bootstrap approach. If we believe, as seems to be the case for the bulk coupling constant, that the boundary coupling constant receives no further corrections at higher orders in perturbation theory, then the formulae containing  $k_r$  and  $b_r$  would be exact.

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